Semi-classical effective equations for isotropic cosmology

Suddhasattwa Brahma

Institute for Gravitation and the Cosmos
Pennsylvania State University

Work with Martin Bojowald and Elliot Nelson
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Semi-classical Effective Equations

- We do not require the exact structure of inner products on the Hilbert space.
- Solve for the moments directly, which are the useful quantities for semi-classical evolution.
- For isotropic and homogeneous cosmology, nature of quantum corrections may be realised, without the technical difficulties of quantizing gravity.
- There is a natural way to recover the classical behaviour known from General Relativity.
- Systematic way to get higher time derivatives in the equations of motion for a canonically quantized system.

Higher Curvature Actions

Quadratic Gravity Action given by

$$S = \frac{1}{16\pi G}\int_M \mathrm{d}^4x\sqrt{-g}\left[R + \alpha R^2 + \beta R^{\mu\nu}R_{\mu\nu} - 2\Lambda\right] \tag{2.1}$$

Closed, FLRW metric is given by

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a(t)^2\left(\frac{\mathrm{d}r^2}{1-r^2} + r^2\mathrm{d}\Omega^2\right) \tag{2.2}$$

The equation of state for radiation is given by

$$p = \frac{1}{3}\rho \tag{2.3}$$

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Equation of Motion

The equation of motion is

$$6A + 6H^2 + 6C - 4\Lambda + 18\,\alpha\left[2\frac{\ddddot{a}}{a} + 8\frac{\dddot{a}}{a}H + 4H^4 + 20H^2C - 8H^2A + 2A^2 - C^2 - 4AC\right] + \frac{3}{2}\beta\left[-2\frac{\ddddot{a}}{a} + 8\frac{\dddot{a}}{a}H - 12H^4 - 184H^2C + 12H^2A + 10A^2 - 48C^2 - 16AC\right] = 0 \qquad (2.4)$$

where $A = \ddot{a}/a$, $H = \dot{a}/a$ and $C = 1/a^2$.

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What does this tell us?

- This equation of motion has higher time derivatives of the scale factor in it than in the classical case
  $\Rightarrow$ Require a systematic way to get higher time derivatives in the quantized theory.
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The New Variables

[Martin Bojowald and Aureliano Skirzewski, 2006]

- Define expectation values, with respect to some state, as:

$$\tilde{G}^{a,n} := \langle (\hat{p} - \langle \hat{p} \rangle)^a (\hat{q} - \langle \hat{q} \rangle)^{n-a} \rangle_{\mathrm{Weyl}} \quad (3.1)$$

- Begin with a Hamiltonian operator: $\widehat{H} = \widehat{H}(\hat{q}, \hat{p})$
  Take its expectation value with respect to the same state to define an 'effective' Quantum Hamiltonian

$$H_Q := \langle \widehat{H} \rangle = \left\langle \widehat{H}(\langle \hat{q} \rangle + (\hat{q} - \langle \hat{q} \rangle), \langle \hat{p} \rangle + (\hat{p} - \langle \hat{q} \rangle)) \right\rangle$$
$$= \sum_{n=0}^{\infty} \sum_{a=0}^{n} \frac{1}{n!} \binom{n}{a} \frac{\partial^n H(q,p)}{\partial p^a \partial q^{n-a}} \tilde{G}^{a,n} \quad (3.2)$$

- A point in this infinite dimensional space is completely specified by $(\langle \hat{q} \rangle, \langle \hat{p} \rangle, \tilde{G}^{a,n})$

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The Poisson Bracket

- Define Possion Bracket as:

$$\{\langle\hat{F}\rangle,\langle\hat{K}\rangle\}=\frac{1}{i\hbar}\left\langle[\hat{F},\hat{K}]\right\rangle \tag{3.3}$$

- Using (3.3), we have:

$$\{\langle\hat{q}\rangle,\langle\hat{p}\rangle\}=1$$
$$\{\langle\hat{q}\rangle,\langle\hat{q}\rangle\}=0=\{\langle\hat{p}\rangle,\langle\hat{p}\rangle\}$$
$$\{\langle\hat{p}\rangle,\tilde{G}^{a,n}\}=0=\{\langle\hat{q}\rangle,\tilde{G}^{a,n}\}$$

$$\left\{\tilde{G}^{a,n},\tilde{G}^{b,m}\right\}=\sum_{r=0}^{\infty}\left[\left(\tfrac{\hbar}{2}\right)^{2r}K[a,b,m,n,r]\tilde{G}^{a+b-2r-1,m+n-4r-2}\right]$$
$$-b(n-a)\tilde{G}^{a,n-1}\tilde{G}^{b-1,m-1}+a(m-b)\tilde{G}^{b,m-1}\tilde{G}^{a-1,n-1}$$

where

$$K[a,b,m,n,r]=\sum_{0\leq f\leq 2r+1}(-)^{r+f}(f!(2r+1-f)!)^{-1}\binom{a}{f}\binom{n-a}{2r+1-f}\binom{b}{f}\binom{m-b}{2r+1-f}.$$

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The Equations of Motion

Let $x := \langle \hat{x} \rangle$ and $p := \langle \hat{p} \rangle$.
The Hamilton's equations of motion gives us

$$\dot{x} = \{x, H_Q\} \tag{3.5}$$
$$\dot{p} = \{p, H_Q\} \tag{3.6}$$
$$\dot{\tilde{G}}^{a,n} = \{\tilde{G}^{a,n}, H_Q\} \tag{3.7}$$

Instead of solving the Schrödinger's partial differential equation, we have to solve this infinite set of coupled ordinary differential equations.

- The validity of the solutions to these equations of motion are subject to certain 'Uncertainty Relations', imposed on the moments.

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The Effective Quantum Hamiltonian

The Hamiltonian for an oscillator with a perturbation term is

$$\widehat{H} = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{q}^2 + \widehat{U}(\hat{q})$$

The corresponding 'effective' Quantum Hamiltonian is

$$H_Q = \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2 q^2 + U(q) + \frac{\hbar\omega}{2}(G^{0,2} + G^{2,2}) + \sum_n \frac{1}{n!}(\hbar/m\omega)^{n/2}U^{(n)}(q)G^{0,n} \qquad (4.1)$$

where $G^{a,n} = \hbar^{-n/2}(m\omega)^{n/2-a}\tilde{G}^{a,n}$ are now dimensionless quantities.

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The equations of motion generated by the effective Quantum Hamiltonian are:

$$\dot{q} = m^{-1}p$$

$$\dot{p} = -m\omega^2 q - U'(q) - \sum_n \frac{1}{n!}(m^{-1}\omega^{-1}\hbar)^{n/2}U^{(n+1)}(q)G^{0,n}$$

$$\begin{aligned}\dot{G}^{a,n} = & -a\omega G^{a-1,n} + (n-a)\omega G^{a+1,n} - \frac{aU''}{m\omega}G^{a-1,n} \\ & + \frac{\sqrt{\hbar}aU'''(q)}{2(m\omega)^{\frac{3}{2}}}G^{a-1,n-1}G^{0,2} + \frac{\hbar aU''''(q)}{3!(m\omega)^2}G^{a-1,n-1}G^{0,3} \\ & - \frac{a}{2}\left(\frac{\sqrt{\hbar}U'''(q)}{(m\omega)^{\frac{3}{2}}}G^{a-1,n+1} + \frac{\hbar U''''(q)}{3(m\omega)^2}G^{a-1,n+2}\right) \\ & + \frac{a(a-1)(a-2)}{3\cdot 2^3}\left(\frac{\sqrt{\hbar}U'''(q)}{(m\omega)^{\frac{3}{2}}}G^{a-3,n-3} + \frac{\hbar U''''(q)}{(m\omega)^2}G^{a-3,n-2}\right) + \cdots \end{aligned} \tag{4.2}$$

We need to make two approximations:

- Moments need to be solved perturbatively in $\left(\frac{\hbar}{L}\right)^{1/2}$. Here L is some angular momentum scale provided by the perturbing potential.
- Need to make an adiabatic approximation for the moments where we assume they are slowly varying with time but the evolution of q and p are free. Derivatives with respect to time in equations of motion are rescaled as $\frac{d}{dt} \rightarrow \lambda \frac{d}{dt}$. In the end, we shall set $\lambda = 1$

Thus, we can expand the moments as

$$G^{a,n} = \sum_e \sum_i G^{a,n}_{e,i} \left(\frac{\hbar}{L}\right)^{e/2} \lambda^i \qquad (4.3)$$

At a given order in $\sqrt{\frac{\hbar}{L}}$, denoted by the index e, the adiabatic approximation gives

$$0 = \left\{ G^{a,n}_{e,0}, H_Q \right\} \qquad (4.4)$$

to leading order, and

$$\dot{G}^{a,n}_{e,i} = \left\{ G^{a,n}_{e,i+1}, H_Q \right\} \qquad (4.5)$$

for higher orders.

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General Procedure

- Get equations for each order in the semi-classical and adiabatic expansions for the moments.
- Break up non-linear terms as (for the first adiabatic order):

$$G^{0,2}G^{a-1,n-1} = G^{0,2}_{0,0}G^{a-1,n-1}_{0,1} + G^{0,2}_{0,1}G^{a-1,n-1}_{0,0}$$

- Each of these equations will have four solutions given by the choices of odd or even a and n.
- Some of these equations are subject to constraints coming from the next order equation in λ.
- Solve for the moments order by order, in both $\left(\frac{\hbar}{L}\right)^{1/2}$ and λ.
- Finally, plug the solutions for the moments (up to some finite order) in the equation of motion for q for a semi-classical trajectory for q.

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- Each of these equations will have four solutions given by the choices of odd or even a and n.
- Some of these equations are subject to constraints coming from the next order equation in λ.
- Solve for the moments order by order, in both $\left(\frac{\hbar}{L}\right)^{1/2}$ and λ.
- Finally, plug the solutions for the moments (up to some finite order) in the equation of motion for q for a semi-classical trajectory for q.

$O(\hbar^0, \lambda^0)$

The equation is:

$$0 = -a\omega G_{0,0}^{a-1,n} + (n-a)\omega G_{0,0}^{a+1,n} - \frac{U''(q)a}{m\omega} G_{0,0}^{a-1,n} \tag{4.6}$$

subject to the constraint (coming from the first order equation) :

$$\frac{1}{\omega} \sum_{a \in \text{even}} \begin{pmatrix} n/2 \\ a/2 \end{pmatrix} \left(1 + \frac{U''(q)}{m\omega^2}\right)^{(n-a)/2} \dot{G}_0^{a,n} = 0 \tag{4.7}$$

which gives the solution

$$G_{0,0}^{a,n} = \frac{(n-a)!a!}{2^n((n-a)/2)!(a/2)!} \left(1 + \frac{U''(q)}{m\omega^2}\right)^{(2a-n)/4} \tag{4.8}$$

for even a and n, and $G_{0,0}^{a,n} = 0$ for odd a and/or n.

- The numerical constant chosen here is such that our expectation values are about the ground state of the harmonic oscillator.

$O(\hbar^0, \lambda^1)$

The solutions are:

$$G^{a,n}_{0,1} = 0 \quad \text{for odd } n$$

$$G^{a,n}_{0,1} = 0 \quad \text{for even } a \text{ and } n$$ (once again to match with the ground state)

$$G^{a,n}_{0,1} = C_{a,n} \frac{U'''(q)\dot{q}}{m\omega^3} \left(1 + \frac{U''(q)}{m\omega^2}\right)^{\frac{2a-n-6}{4}} \quad \text{for odd } a \text{ and even } n$$

where $C_{a,n}$ is a dimensionless prefactor given by:

$$C_{a-1,n} = -\frac{(n-a)!(a-1)!}{2^{n+2}(\frac{n-a}{2})!(\frac{a}{2})!}(2a-n) - 2^{-n-2} \sum_{b=0}^{\frac{n-a-2}{2}} \left[\prod_{c=0}^{b} \frac{n-(a+2c)}{a+2c}\right] \frac{(n-a')!(a'-1)!}{(\frac{n-a'}{2})!(\frac{a'}{2})!}(2a'-n)$$

for even a, where $a' = a + 2(b+1)$.

$O(\hbar^1, \lambda^0)$

The solutions are:

$$G^{a,n}_{1,0} = 0 \text{ for odd } a$$

$$G^{a,n}_{1,0} = 0 \text{ for even } a \text{ and } n \text{ (vacuum state considerations)}$$

$$G^{a,n}_{1,0} = \sqrt{L}\, D_{a,n} \frac{U'''(q)}{m^{3/2}\omega^{5/2}} \left(1 + \frac{U''(q)}{m\omega^2}\right)^{\frac{2a-n-5}{4}} \quad \text{for even } a \text{ and odd } n$$

where

$$D_{a,n} = \begin{cases} \frac{(-1)^b \Gamma\left(\frac{n}{2}\right)}{12\pi(1-\frac{n}{2})_b}\Big((n-1)b!\sqrt{\pi} + (n-8b-1)\Gamma\left(b+\frac{1}{2}\right) \\ \quad - \sum_{c=0}^{b-2}(-1)^c(n-8(b-c-1)-1)\Gamma\left(b-c-\frac{1}{2}\right)(-b)_{c+1}\Big) & \text{if } n \geq 5,\ b \geq 2 \\ \frac{n-1}{12\pi}\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{1}{2}\right) & \text{if } n \geq 3,\ b = 0 \\ \frac{3n-11}{12\pi(n-2)}\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{1}{2}\right) & \text{if } n \geq 3,\ b = 1 \end{cases}$$

is a dimensionless prefactor that depends on a and n. In the above expression, $b = (n-a-1)/2$ and $(x)_n = x(x+1)...(x+n-1)$ is the Pochhammer symbol.

Equation of motion for q is thus:

$$\ddot{q} = -\omega^2 q - U'(q)/m - \frac{\hbar}{2m^2\omega} U'''(q) \left[\sum_{\lambda=0}^{4} G^{0,2}_{0,\lambda} + \left(\frac{\hbar}{L}\right)^{1/2} \sum_{\lambda=0}^{4} G^{0,2}_{1,\lambda} \right] \quad (4.9)$$

where the relevant moments are

$$G^{0,2}_{0,0} = \frac{1}{2}\left(1 + \frac{U''(q)}{m\omega^2}\right)^{-\frac{1}{2}}$$

$$G^{0,2}_{0,2} = \frac{U'''(q)\ddot{q} + U''''(q)\dot{q}^2}{16m\omega^4}\left(1 + \frac{U''(q)}{m\omega^2}\right)^{-\frac{5}{2}} - \frac{5(U'''(q)\dot{q})^2}{64m^2\omega^6}\left(1 + \frac{U''(q)}{m\omega^2}\right)^{-\frac{7}{2}}$$

$$\begin{aligned} G^{0,2}_{0,4} = & -\frac{U'''(q)\ddddot{q} + 4U''''(q)\dddot{q}\dot{q} + 3U''''(q)\ddot{q}^2 + 6U'''''(q)\dot{q}^2\ddot{q} + U''''''(q)\dot{q}^4}{64m\omega^6}\left(1 + \frac{U''(q)}{m\omega^2}\right)^{-\frac{7}{2}} \\ & + \left[\frac{21(U''''(q)\dot{q}^2 + U'''(q)\ddot{q})^2}{256m^2\omega^8} + \frac{7(U'''(q)\dot{q})(U'''(q)\dddot{q} + 3U''''\ddot{q}\dot{q} + U'''''(q)\dot{q}^3)}{64m^2\omega^8}\right]\left(1 + \frac{U''(q)}{m\omega^2}\right)^{-\frac{9}{2}} \\ & - \frac{231(U'''(q)\ddot{q} + U''''(q)\dot{q}^2)(U'''\dot{q})^2}{512m^3\omega^{10}}\left(1 + \frac{U''(q)}{m\omega^2}\right)^{-\frac{11}{2}} \\ & + \frac{1155(U'''(q)\dot{q})^4}{4096m^4\omega^{12}}\left(1 + \frac{U''(q)}{m\omega^2}\right)^{-\frac{13}{2}} \end{aligned}$$

Equation of motion for q up to $\hbar^{3/2}$ and fourth adiabatic order

We may now rewrite the equation of motion as:

$$\begin{aligned}\ddot{q} &= -\omega^2 q - U'(q)/m \\ &\quad - \frac{\hbar}{2m^2\omega} U'''(q)\left[f(q,\dot{q}) + f_1(q,\dot{q})\ddot{q} + f_2(q)\ddot{q}^2 + f_3(q,\dot{q})\dddot{q} + f_4(q)\ddddot{q}\right]\end{aligned} \tag{4.10}$$

where

$$\begin{aligned}f(q,\dot{q}) &= \frac{1}{2}\left(1+\frac{U''(q)}{m\omega^2}\right)^{-1/2} + \frac{U''''(q)\dot{q}^2}{16m\omega^4}\left(1+\frac{U''(q)}{m\omega^2}\right)^{-5/2} - \frac{5(U'''(q))^2\dot{q}^2}{64m^2\omega^6}\left(1+\frac{U''(q)}{m\omega^2}\right)^{-7/2} \\ &\quad - \frac{U''''''(q)\dot{q}^4}{64m\omega^6}\left(1+\frac{U''(q)}{m\omega^2}\right)^{-7/2} + \frac{21(U''''(q))^2\dot{q}^4}{256m^2\omega^8}\left(1+\frac{U''(q)}{m\omega^2}\right)^{-9/2} \\ &\quad + \frac{7U'''''(q)U'''(q)\dot{q}^4}{64m^2\omega^8}\left(1+\frac{U''(q)}{m\omega^2}\right)^{-9/2} - \frac{231U''''(q)(U'''(q))^2\dot{q}^4}{512m^3\omega^{10}}\left(1+\frac{U''(q)}{m\omega^2}\right)^{-11/2} \\ &\quad + \frac{1155(U'''(q))^4\dot{q}^4}{4096m^4\omega^{12}}\left(1+\frac{U''(q)}{m\omega^2}\right)^{-13/2}\end{aligned} \tag{4.11}$$

$$f_1(q,\dot{q}) = \frac{U'''(q)}{16m\omega^4}\left(1+\frac{U''(q)}{m\omega^2}\right)^{-5/2} - \frac{3U'''''(q)\dot{q}^2}{32m\omega^6}\left(1+\frac{U''(q)}{m\omega^2}\right)^{-7/2} + \frac{63U''''(q)U'''(q)\dot{q}^2}{128m^2\omega^8}\left(1+\frac{U''(q)}{m\omega^2}\right)^{-9/2} - \frac{231(U'''(q))^3\dot{q}^2}{512m^3\omega^{10}}\left(1+\frac{U''(q)}{m\omega^2}\right)^{-11/2} \quad (4.12)$$

$$f_2(q) = -\frac{3U''''(q)}{64m\omega^6}\left(1+\frac{U''(q)}{m\omega^2}\right)^{-7/2} + \frac{21(U'''(q))^2}{256m^2\omega^8}\left(1+\frac{U''(q)}{m\omega^2}\right)^{-9/2} \quad (4.13)$$

$$f_3(q,\dot{q}) = -\frac{U''''(q)\dot{q}}{16m\omega^6}\left(1+\frac{U''(q)}{m\omega^2}\right)^{-7/2} + \frac{7(U'''(q))^2\dot{q}}{64m^2\omega^8}\left(1+\frac{U''(q)}{m\omega^2}\right)^{-9/2} \quad (4.14)$$

$$f_4(q) = -\frac{U'''(q)}{64m\omega^6}\left(1+\frac{U''(q)}{m\omega^2}\right)^{-7/2} \quad (4.15)$$

Isotropic and Homogeneous Cosmology

Starting with the Einstein-Hilbert action (with the FLRW metric), including a cosmological constant and matter, we can write the Friedmann Equation as (setting $\frac{8\pi G}{3} = 1$):

$$\frac{1}{4}\frac{p_a^2}{a^4} + \frac{k}{a^2} - \frac{\Lambda}{3} = \rho \tag{5.1}$$

where a is the scale factor, Λ is the cosmological constant, ρ is the energy density, $p_a = -\frac{2a\dot{a}}{N}$ is the momentum canonically conjugate to a and N is the usual lapse function.

For a closed universe $(k = 1)$ and the radiation-dominated era $\rho = \frac{p_t}{a^4}$, we have a Hamiltonian which generates evolution with respect to some time co-ordinate t, related to the proper time τ as $t = \int_0^{\tau} a(\tau')^{-1}\mathrm{d}\tau'$, given by:

$$H = p_t = \frac{1}{4}p_a^2 + a^2 - \frac{\Lambda}{3}a^4 \tag{5.2}$$

So we have an Anharmonic Oscillator Hamiltonian with $m = 2$, $\omega = 1$ and $U(a) = -\frac{\Lambda}{3}a^4$.

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Numerical Solutions for the scale factor

We choose $\Lambda = 0.3$ and and assume the ansatz to be of the form $a(t) = a_0(t) + \hbar a_1(t) + \cdots$

The solutions are:

- very sensitive to initial conditions, which are only specified for the classical solution. The quantum corrections are all assumed to have zero initial conditions,
- very sensitive to the strength of the perturbing potential, which should be close to the vacuum state of the harmonic oscillator,
- helpful in understanding where the quantum corrections might possibly become important.

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Figure: Solution for a_0

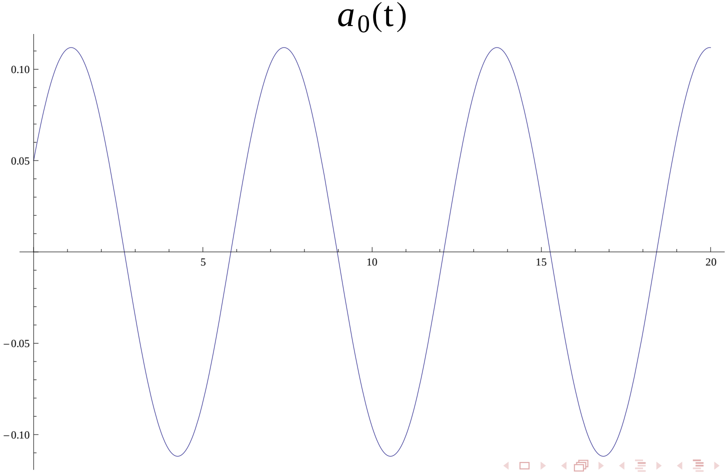


Figure: Solution for a_1

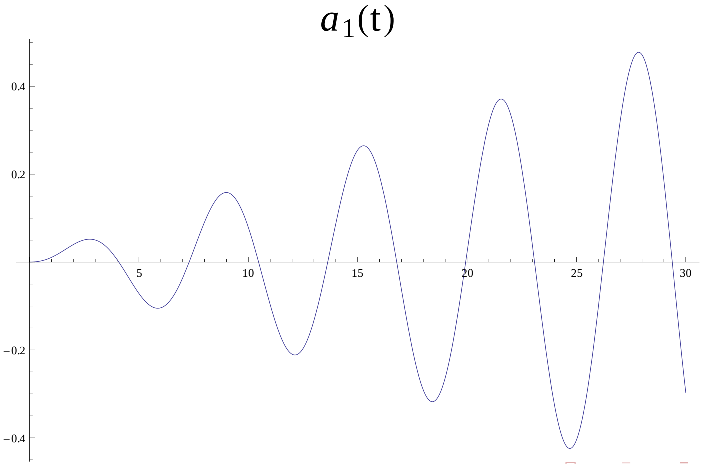
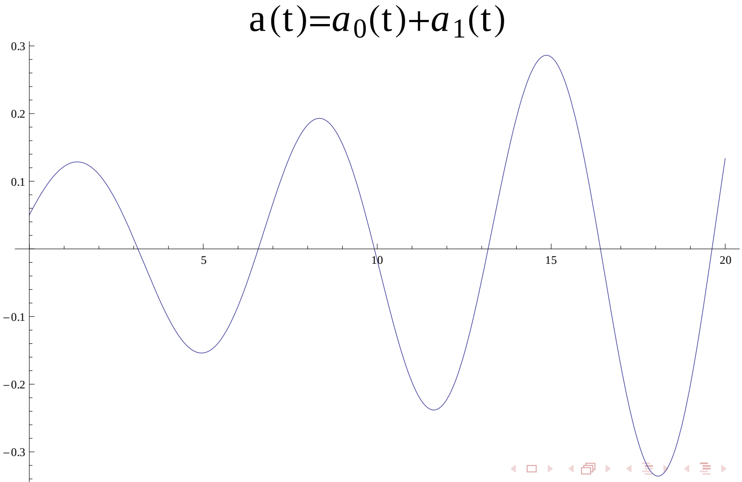


Figure: Solution for a



Results

- The quantum corrections to the scale factor prevents it from going back to zero where the classical solution did!!! Although the quantum corrections are small usually, they play a significant role when the classical solution goes to zero. This result indicates that the scale factor may be saved (or, at least, delayed) from going back to the singular point in the presence of quantum corrections.
- The acceleration from the classical solution is negative for the first half cycle (as expected during the radiation-dominated era). However the acceleration for the overall scale factor (including quantum corrections) turns positive at some points in this period. This also indicates that this positive acceleration, coming from the quantum corrections, may drive the scale factor away from zero!!

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Important lessons and looking ahead

- The quantum back-reactions from the moments might be able to save the scale factor from space-like singularities even if the classical solution starts from zero.
- The behaviour of the scale factor from this analysis should, at least numerically, be similar to that from higher curvature actions (e.g. quadratic gravity).
- Apply similar methods to find 'effective equations' for quantum field theory. An important application of that would be understanding quantum corrections to the inflaton field.

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