Algebra basic exam, January 2024

180 minutes

1. (10 points)

   (a) Show that an integral domain with finitely many elements is a field.

   (b) Show that the multiplicative group of a finite field is cyclic.

2. Part I. (10 points)

   Let G be a finitely generated group with the generator set $g_1, \dots, g_m$. Let H be a subgroup of G of finite index. Let $Hx_1, \dots, Hx_n$ be all the right cosets of H, with $x_1 = 1$.

   (a) Show that, for each i, j, there is $h_{i,j} \in H$ satisfying $x_i g_j = h_{i,j} x_r$ for some $r = r(i, j)$.

   (b) Show that the group H is finitely generated by proving that the set $\{h_{i,j} : i = 1, 2, \dots, n,\ j = 1, 2, \dots, m\}$ generates H.

   Part II. (10 points)

   Recall that a group is *torsion* if every element of the group has finite order. Let G be a finitely generated solvable torsion group. Prove that G is finite. [Hint: start by proving that abelianization of G is finite. Then use Part I. You may assume Part I even if you did not solve it.]

3. (10 points) Let F be a subfield of $\overline{Q}$ that is maximal among all subfields not containing $\sqrt{2}$. Let E/F be a Galois extension of F with $E \neq F$.

   (a) Prove that $F(\sqrt{2})$ is the unique subfield of E that is a degree-2 extension of F.

   (b) Prove that $\mathrm{Gal}(E/F)$ is cyclic.

   [You do not need to prove the existence of fields E and F satisfying the conditions in this problem.]

4. (10 points)

   (a) Define the term “Noetherian ring”

   (b) Let R be a Noetherian ring, and let S be a multiplicative subset of R. Show that the ring $S^{-1}R$ is Noetherian.