Discrete Mathematics: Basic Exam

January 19, 2024

Do not flip the page until instructed.

Name: ______________________________

Problem	Points
1	
2	
3	
4	
5	
Total	

Each problem is worth 20 points.

Problem 1 **20 points**

Let G be a random graph sampled according to $G(n, \frac{1}{2})$, that is, G has n vertices and each edge is present independently with probability $\frac{1}{2}$. For $\varepsilon > 0$, let $k = (2+\varepsilon)\log_2(n)$. Show that every independent set in G has size at most k with high probability.

Problem 2 **20 points**

Let $X \subset \mathbb{R}^2$ be a set of n points. Show that there is a red/blue coloring of the points in X such that in every axis-parallel rectangle R the number of red points deviates from the number of blue points by at most $O(\sqrt{n \log n})$.

Problem 3 **20 points**

Let $k \geq 1$ and $q \geq 2$ be integers, and let $n = kq + 2$. Show that every sufficiently large finite $X \subseteq \mathbb{R}^2$ in general position contains a subset $Y \subseteq X$ of size n such that Y is in convex position and the number of points of X that are in the interior of conv Y is divisible by q.

Recall that conv Y is the smallest convex polygon that contains Y.

Problem 4 **20 points**

Let F be a family of subsets of $[n]$ that does not contain three pairwise distinct sets $A, B, C \in F$ with $A \subset B \subset C$. Show that $|F| \leq 2\binom{n}{\lceil n/2 \rceil}$ and that this is optimal for odd n.

Problem 5 **20 points**

Let P be a lattice d-polytope. Show that $(d+1) \cdot P$ contains an interior lattice point. For every dimension d give an example of a lattice d-polytope P that shows that $d \cdot P$ does not necessarily contain an interior lattice point.