Department of Mathematical Sciences
Carnegie Mellon University

Basic Examination
Introduction To Functional Analysis
January 2024

Time allowed: 180 minutes.

This test is closed book: no notes or other aids are permitted. You may use without proof standard results from the syllabus which are independent of the question asked, unless explicitly instructed otherwise. You must, however, clearly state the result you are using.

1. State and prove the Riesz representation theorem in Hilbert spaces.

2. Let X be a Banach space and let $\Pi : X \to X$ be a linear function such that $\Pi(X)$ and $\Pi^{-1}(\{0\})$ are closed and

$$\Pi(\Pi(x)) = \Pi(x) \text{ for all } x \in X.$$

   (a) Prove that the graph of Π,

$$\operatorname{gr}\Pi = \{(x, \Pi(x)) : x \in X\},$$

   is closed.

   (b) Prove that Π is continuous.

3. Let X be an infinite dimensional Banach space.

   (a) Prove that if $n \in \mathbb{N}$, $L_1, \ldots, L_n \in X'$, and $x_1, \ldots, x_{n+1} \in X$ are linearly independent, then there exists a linear combination x_0 of $x_1, \ldots, x_{n+1}$ such that $x_0 \neq 0$ and $L_i(x_0) = 0$ for all $i = 1, \ldots, n$.

   (b) Use part (a) to prove that the open ball $B(0, 1)$ is not weakly open.

4. Let $X = C([0, 1])$ with the norm

$$\|f\| = \max_{x \in [0,1]} |f(x)|,$$

   and let $T : X \to X$ be defined by

$$T(f)(x) = f(x^2).$$

   (a) Prove that $T(Y)$ is continuous and determine its norm.

   (b) Is T compact?