Basic Examination: Measure and Integration January 2016

- This test is **closed book**: no notes or other aids are permitted.
- You have 2 hours. The exam has a total of 4 questions and 100 points (25 each).
- You may use without proof *standard* results from the syllabus which are independent of the question asked, unless explicitly instructed otherwise. You must, however, **clearly** state the result you are using.

Below, if not stated explicitly, $(X, \mathcal{F}, \mu)$ is a measure space, $L_p = L_p(X, \mathcal{F}, \mu)$ is the standard L_p space ($p \in [1, \infty]$) and $C_c(X)$ is the space of continuous functions from X to $\mathbb{R}$ having compact support. Moreover, m is Lebesgue measure, $\mathcal{L}$ is the σ-algebra of Lebesgue-measurable sets, and $\mathcal{B}(X)$ is the Borel σ-algebra of subsets of X.

1. (a) (True or false) Prove or provide a counterexample:

If U is a subset of $\mathbb{R}$ which is open and dense, then $m(U) = +\infty$.

(b) Prove that for every Lebesgue measurable set $E \subset \mathbb{R}$ there is a Borel set F such that the symmetric difference $(E \setminus F) \cup (F \setminus E)$ has Lebesgue measure zero.

2. Let $d > 1$ and let $z \in \mathbb{R}^d$ with $z \neq 0$. For any $u \in C_c(\mathbb{R}^d)$, define

$$(A_z u)(x) = \int_{[0,1]} u(x + tz)\, dm(t), \qquad x \in \mathbb{R}^d.$$

Prove that the restriction of A_z from $C_c(\mathbb{R}^d)$ to $C_c(\mathbb{R}^d) \cap L_p(\mathbb{R}^d)$ has a unique continuous extension

$$\hat{A}_z : L_p(\mathbb{R}^d) \to L_p(\mathbb{R}^d).$$

(Remark: The formula above *does not* define a function $A_z u : \mathbb{R}^d \to \mathbb{R}$ for all Lebesgue integrable $u : \mathbb{R}^d \to \mathbb{R}$.)

3. A function $f\colon \mathbb{R} \to \mathbb{R}$ is *nonexpansive* if $|f(x) - f(y)| \leq |x - y|$ for all $x, y \in \mathbb{R}$. Prove that f is nonexpansive if and only if f is absolutely continuous and $|f'(x)| \leq 1$ a.e.

4. Let μ and ν be finite Radon measures on $X = \mathbb{R}^d$.

(a) Assuming ν is *not* absolutely continuous with respect to μ, show that there exists a sequence of continuous functions $\phi_n : X \to [0, 1]$ having compact support, such that

$$\int \phi_n \, d\mu \Big/ \int \phi_n \, d\nu \to 0 \quad \text{as } n \to \infty.$$

(b) Define

$$K(\mu, \nu) = \sup \left\{ \int f \, d\mu + \int g \, d\nu : f, g \in C_c(X) \text{ and } f(x) + \tfrac{1}{2} g(x)^2 \leq 0 \ \forall x \right\}.$$

Prove that if $K(\mu, \nu) < +\infty$ then $\nu \ll \mu$. (Hint: use $f = a_n \phi_n$, $g = b_n \phi_n$.)

(Remark: using the result of this problem one can show $K(\mu, \nu) = \int_{\mathbb{R}^d} \frac{1}{2} |u|^2 \, d\mu$, $u = \frac{d\nu}{d\mu}$.)