Basic Examination: MEASURE AND INTEGRATION
January 16, 2024

• This test is closed book: no notes, Internet sources, or other aids are permitted.

• You have 3 hours. The exam has a total of 4 questions and 76 points.

• You may use without proof standard results from the syllabus which are independent of the question asked. You must, however, clearly state the result you are using.

**1.**

(i) [6 points] State Lebesgue Monotone Convergence Theorem.

(ii) [10 points] State and prove Fatou's Lemma.

**2.** [12 points] Let $f \in L^p_{\text{loc}}(\mathbb{R}^N)$, and let $1 \leq p < \infty$. Prove that for $\mathcal{L}^N$ a.e. $x \in \mathbb{R}^N$,

$$\lim_{r \to 0} \frac{1}{\mathcal{L}^N(B(x,r))} \int_{B(x,r)} |f(y) - f(x)|^p \, dy = 0,$$

where $\mathcal{L}^N$ stands for the Lebesgue measure on $\mathbb{R}^N$.

**3.** [24 points] Let $(X, \mathcal{M})$ be a measurable space, let $\mu$ be a $\sigma$-finite nonnegative measure on $(X, \mathcal{M})$, and let $\lambda$ be a real-valued measure on $(X, \mathcal{M})$. Recall that the total (nonnegative) variation measure $|\lambda|$ is defined by

$$\begin{aligned} |\lambda|(E) := \quad & \sup \{ |\lambda(E_1)| + \ldots |\lambda(E_n)| : n \in \mathbb{N}, \\ & E_i, i = 1 \ldots n, \text{are disjoint sets in } \mathcal{M}, \cup_{i=1}^n E_i \subset E \}. \end{aligned}$$

Prove that the following are equivalent:

(i) $\lambda << \mu$.

(iii) For every $\varepsilon > 0$ there exists $\delta > 0$ such that if $E \in \mathcal{M}$ is such that $\mu(E) < \delta$ then $|\lambda|(E) < \varepsilon$.

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**4.** Let $(X, \mathcal{M}, \mu)$ be a measure space. $\mu$ is said to have the *finite subset property* if for every $E \in \mathcal{M}$ with $\mu(E) > 0$ there exists $F \subseteq E$ such that $0 < \mu(F) < \infty$.

(i) [12 points] Prove that $\mu$ satisfies the finite subset property if and only if for every $E \in \mathcal{M}$ with $\mu(E) > 0$

$$\mu(E) = \sup\{\mu(F) : F \in \mathcal{M}, F \subseteq E, 0 < \mu(F) < \infty\}.$$

(ii) [12 points] Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be measure spaces with $\mu$ and $\nu$ non zero. Prove that if $\mu \times \nu$ satisfies the finite subset property, then $\mu$ and $\nu$ also satisfy the finite subset property.