Basic examination: probability

January 26th, 2024

The exam is 180 minutes long

Problem 1 (10pts). *Give definitions of*

- *Sigma field*
- *Convergence of a sequence of random variables* $(X_n)_{n=1}^{\infty}$ *to a random variable* X **in probability**
- *Discrete time martingale*
- $\pi-$ *and* $\lambda-$*systems*
- *Mutual independence of events*

Problem 2 (10pts). *State the following:*

- *The monotone convergence theorem*
- *Markov/Chebyshev inequality*
- *Jensen's inequality*
- *The Strong Law of Large Numbers for a sequence of i.i.d random variables*
- *The Fubini–Tonelli theorem*

Problem 3 (20pts).

- *Let* $X_0, X_1, \dots, X_n$ *be a martingale with* $X_0 = 0$ *such that the martingale differences* $X_m - X_{m-1}$, $1 \le m \le n$, *are bounded random variables. Use the definition of martingale to prove that the variance of* X_n *equals the sum of the variances of the martingale differences.*
- *Assume that* Y_i, $i \ge 1$, *are symmetric sign variables, that is,* $\mathbb{P}\{Y_i = 1\} = \mathbb{P}\{Y_i = -1\} = 1/2$ *for all* i. *Assume further that the sequence of partial sums* $\sum_{i=1}^{n} Y_i$, $n \ge 0$, *is a martingale. Use the definition of martingale to prove that* Y_i, $i \ge 1$, *are mutually independent.*
- *Let* $b_1, b_2, \dots, b_n$ *be mutually independent Bernoulli(1/2) variables, and let* $f : \mathbb{R}^n \to \mathbb{R}$ *be a 1–Lipschitz function, that is,* $|f(x) - f(y)| \le \|x - y\|_2$ *for all* $x, y \in \mathbb{R}^n$. *Define the sequence*

$$X_0 := \mathbb{E}\, f(b_1, \dots, b_n),$$
$$X_i := 2^{-n+i} \sum_{s_1,\dots,s_{n-i} \in \{0,1\}} f(s_1, \dots, s_{n-i}, b_{n-i+1}, \dots, b_n), \quad 1 \le i \le n.$$

Prove that this is a martingale and that the martingale differences $X_i - X_{i-1}$, $1 \le i \le n$, *are bounded by 1 almost surely.*

Problem 4 (15pts). *State the Lévy continuity theorem for characteristic functions. Apply the result to derive the Central Limit Theorem for i.i.d random variables.*

Problem 5 (15pts). *Let* X *be a non-negative random variable with* $\mathbb{E}\, X < \infty$. *Prove that there exists a convex function* $\phi : [0, \infty) \to [0, \infty)$ *with* $\lim\limits_{t\to\infty} \frac{\phi(t)}{t} = \infty$, *such that* $\mathbb{E}\, \phi(X) < \infty$.

Problem 6 (10pts)**.** *For each $n \geq 1$, let $b_{n1}, b_{n2}, \ldots, b_{nn}$ be i.i.d Bernoulli(p_n) variables, where the sequence of positive numbers p_n, $n \geq 1$, satisfies $\lim_{n\to\infty}(np_n) = \lambda > 0$. Prove that the sequence of variables*

$$\sum_{i=1}^{n} b_{ni}, \quad n \geq 1,$$

converges weakly to the Poisson distribution with parameter λ.

Problem 7 (10pts)**.** *Let $L > 1$ be a fixed integer. Let $b_1, b_2, \ldots$ be a sequence of mutually independent Bernoulli(1/2) variables. Define a sequence of discrete random variables $X_0, X_1, \ldots$ taking values in $\{0, 1, \ldots, L-1\}$ as*

$$X_n := \Big(\sum_{i=1}^{n} b_i\Big) \mod L.$$

Prove that X_n, $n \geq 1$, converge weakly to the uniform distribution on the set $\{0, 1, \ldots, L-1\}$.

Problem 8 (10pts)**.** *Let $X_1, X_2, \ldots$ be a sequence of i.i.d random variables of zero mean and unit variance. Assume that the sequence*

$$\sum_{k=1}^{n} \frac{1}{2^{k/2}} X_k, \quad n \geq 1,$$

converges weakly to the standard normal variable. Prove that X_i's are standard normal variables.